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A solvable toy model for tachyon condensation in string field theory

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Abstract: The lump solution of ϕ^3 field theory provides a toy model for unstable D-branes of bosonic string theory. The field theory living on this lump is itself a cubic field theory involving a tachyon, two additional scalar fields, and a scalar field continuum. Its action can be written explicitly because the fluctuation spectrum of the lump turns out to be governed by a solvable Schroedinger equation; the $\ell = 3$ case of a series of reflectionless potentials. We study the multiscalar tachyon potential both exactly and in the level expansion, obtaining insight into issues of convergence, branches of the solution space, and the mechanism for removal of states after condensation. In particular we find an interpretation for the puzzling finite domain of definition of string field marginal parameters.

Keywords: Solitons Monopoles and Instantons, Bosonic Strings, D-branes.

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Contents

1. Introduction and summary

It has recently been realized that Sen's conjectures on tachyon condensation and D-brane annihilation [1]–[4] can be studied quite effectively using string field theory (SFT) [5]–[17], [19]–[21]. With strong evidence now available to the physical correctness of the conjectures, we also have the opportunity to use tachyon dynamics to refine our understanding of string field theory. While the level expansion studies of the bosonic D-brane tachyon potential find with high accuracy the requisite critical point, an analytic solution, showing conclusively the existence of this critical point is still lacking. Such solution for the tachyon condensate would surely teach us a lot about the nature of the string field equations and would help finding many other interesting solutions.

One way to make progress with this question is to develop techniques to deal with the full string field equations. These techniques would presumably use the universality of the tachyon potential [22] thus requiring methods where various Virasoro algebras play a central role, as in [23]. Another line of investigation is to use toy models and/or simplified models of the tachyon condensation phenomenon. Particularly interesting have been a study in the framework of p-adic open string theory [24] and papers where non-commutativity is added via a magnetic field on the D-brane [25]–[30].

In this paper we develop a toy model which captures a different aspect of the problem: the decay of a lump as seen by the field theory of the lump itself. Bosonic D-branes are lump solutions of string field theory. The field theory on a D-brane itself is indeed a string field theory, and contains a multiscalar tachyon potential. In finding the critical point of this string field potential we are indeed exploring the decay of a lump from the viewpoint of the field theory living on the world-volume of the lump.

We begin with the simple ϕ^3 theory, which is the truncation of open SFT to the tachyon field only. As analyzed numerically in [10] this field theory has a codimension one lump solution that provides a first approximation to the codimension one brane of string theory. Additionally, being an unstable lump, the field theory living on the lump has again a tachyon with an estimated $m^2 \approx -1.3$ [10]. In fact, the exact lump profile in ϕ^3 theory is readily written in terms of hyperbolic functions. More surprising, however, is that the Schroedinger type equation for fluctuations that determines the spectrum of the field theory on the lump is exactly solvable. The associated Schroedinger potential is actually the $l = 3$ case of the infinite series of exactly solvable reflectionless potentials:[1] $U_\ell(x) = -\ell(\ell+1)\,\mathrm{sech}^2\, x$. In fact, it is known that the fluctuation spectrum of the sine-Gordon soliton is governed by the $\ell = 1$ potential and the the spectrum of the ϕ^4 kink solution is governed by the $\ell = 2$ potential [33]. In both cases the field theory on the soliton has no tachyon since the soliton is stable. If one tries to associate the $\ell = 3$ potential to a stable soliton the result is a very strange field theory potential with branch cuts [33, 34]. Apparently U_3 was not know to be relevant to the unstable lump of the very simple ϕ^3 theory.

This solvability allows us to find the complete field content of the ϕ^3 lump field theory. This includes a tachyon with $m^2 = -5/4$, a massless scalar, a massive scalar with $m^2 = 3/4$, and a continuum spectrum of scalar fields with $m^2 \geq 1$. Given that we know the analytic expressions for the fluctuation eigenfunctions, the exact cubic multiscalar field theory living on the lump can be calculated exactly.

We then make the following claim: *The multiscalar tachyon potential of the unstable lump field theory must have a critical point with a negative energy density equal to the energy density of the lump itself.* Indeed, the field theory of the lump describes the fluctuations of the lump, and one possible fluctuation of the unstable lump is the no-lump configuration. Being a critical point in field space with zero

[1] I am grateful to Jeffrey Goldstone for providing this identification, and teaching me how to solve elegantly for the spectrum of such hamiltonians. For a pedagogical review on these and other solvable hamiltonians with references to the early literature see [31]. Applications of reflectionless systems to fermions can be found in [32].

energy density, this configuration must appear as a lower energy state of the lump field theory. This remark shows that the critical point of the tachyon field theory of the lump is easily found. Let $\overline{\phi}(x)$ represent the profile of the lump itself and let ϕ_0 denote the expectation value of the field at the zero energy vacuum, with $\lim_{x\to\pm\infty}\overline{\phi}(x) = \phi_0$. In addition, let the fluctuation fields around the lump be written as $\sum_n \phi_n\psi_n(x)$ where the $\psi_n(x)$'s are the eigenfunctions of the Schroedinger problem and the ϕ_n's are the fields living on the lump. Then the critical point of the tachyon potential is defined by the equation: $\phi_0 - \overline{\phi}(x) = \sum_n \phi_n\psi_n(x)$ which fixes the ϕ_n's. Namely, *the expectation values for the lump fields at the tachyonic vacuum are given as the expansion coefficients of minus the lump profile in terms of the eigenfunctions of the lump fluctuation equations.*

Equipped with both a level expansion approach to the multiscalar potential and the exact solution for the tachyon condensate, we explore the convergence of the level expansion in this model and find the large level behavior of the expectation values of fields representing the condensate. We are also able to track explicitly the masses of the tachyon and other fields as the lump goes through the annihilation process. The tachyon and other fields on the lump flow and join eventually the continuum spectrum of states associated to the massive field defined on the locally stable vacuum of the ϕ^3 model. These results are consistent with the discussion of ref. [35] of the annihilation of a kink and an anti-kink of ϕ^4 field theory.

Our study of the toy model gives a plausible resolution to the puzzles found by Sen and the author [19] concerning the definition of marginal fields in string field theory. While conformal field theory marginal parameters are naturally defined over infinite ranges, it was found that the effective potential for the string field marginal parameter fails to exist beyond a critical value. This appeared to mean that either SFT compresses the CFT moduli into a compact domain, or that SFT does not cover all of CFT moduli space. By studying the toy model we are led to propose that the map from the CFT marginal moduli to SFT marginal moduli is actually two to one; as the CFT parameter parameter grows from zero to infinity the SFT parameter grows from zero to a maximum value and then decreases back to zero! On the way back, a different solution branch for high level fields must be chosen. This behavior is borne out by the analysis of how the massless state in the lump, whose wavefunction is the derivative of the profile, implements translations of the lump. Indeed, even for large translations the expectation value for this massless state is always bounded.

This paper is organized as follows. The lump solution and the spectrum is presented in section 2. The multiscalar potential for the fields living on the brane is computed in section 3. A detailed analysis of the tachyon condensation and the removal of the lump states by the condensation process is given in section 4. The puzzle of marginal parameters is analyzed in section 5, along with a suggestion for a solution branch describing large marginal deformations. This paper concludes in section 6 with a discussion of the results.

JHEP09(2000)028

2. The ϕ^3 lump and its fluctuation spectrum

In this section we will consider ϕ^3 scalar field theory and find the exact lump profile as well as the exact lump energy. We then turn to the fluctuations around the lump solution, identifying the resulting Schroedinger like equation. This Schroedinger equation turns out to be solvable, both for the discrete and continuum spectrum. We give the explicit energy eigenfunctions thus identifying the spectrum of the field theory living on the lump worldvolume.

2.1 The lump solution

We begin with the ϕ^3 scalar field theory in $p+2$ space-time dimensions with action

$$S = \int dt d^p y dx \left[\frac{1}{2}\Big(\frac{\partial \phi}{\partial t}\Big)^2 - \frac{1}{2}\nabla_{\vec{y}}\phi \cdot \nabla_{\vec{y}}\phi - \frac{1}{2}\Big(\frac{\partial \phi}{\partial x}\Big)^2 - V(\phi)\right], \tag{2.1}$$

where we have separated out the x coordinate, to be used to produce the lump solution. The resulting lump will therefore represent a p brane, with $\vec{y}$ denoting the p spatial coordinates of the brane worldvolume. In the action above, the potential will be taken to be:

$$V(\phi) = \frac{1}{3\phi_*}(\phi-\phi_*)^2\left(\phi + \frac{\phi_*}{2}\right) = -\frac{1}{2}\phi^2 + \frac{1}{3\phi_*}\phi^3 + \frac{\phi_*^2}{6}. \tag{2.2}$$

This is a generic ϕ^3 potential, it has a local maximum and a local minimum rather than an inflection point. The local maximum is at $\phi = 0$, where we have a tachyon of $m^2 = -1$ (this particular value can be thought as a choice of units). The local minimum is at $\phi = \phi_*$, where we have a scalar particle of $m^2 = +1$ (this equality of squared masses up to a sign, is generic). The open string field theory action, truncated to the tachyon and with $\alpha' = 1$, is given as S/g_0^2, where g_0 is the open string coupling constant, and S is the above action with $p = 24$ and $\phi_* = 1/K^3$ with $K = 3\sqrt{3}/4$. By rescaling the field variable as $\phi \to \phi_*\phi$, the action and potential can be simply written as

$$\begin{aligned} S &= (\phi_*)^2 \int d^{p+1}y dx \left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\Big(\frac{\partial\phi}{\partial x}\Big)^2 - V(\phi)\right], \\ V(\phi) &= \frac{1}{3}(\phi-1)^2\Big(\phi+\frac{1}{2}\Big) = \frac{1}{6} - \frac{1}{2}\phi^2 + \frac{1}{3}\phi^3, \end{aligned} \tag{2.3}$$

where ϕ_* appears now as an overall multiplicative constant in the action, and therefore, it does not appear in the equations of motion. All energies will be measured in units of $(\phi_*)^2$, which, for the purposes of the present paper will be set to unity. In the above equation we have defined $y^\mu = (t, \vec{y})$ with metric $(-,+,+,\ldots,+)$. The potential is shown in figure 1.

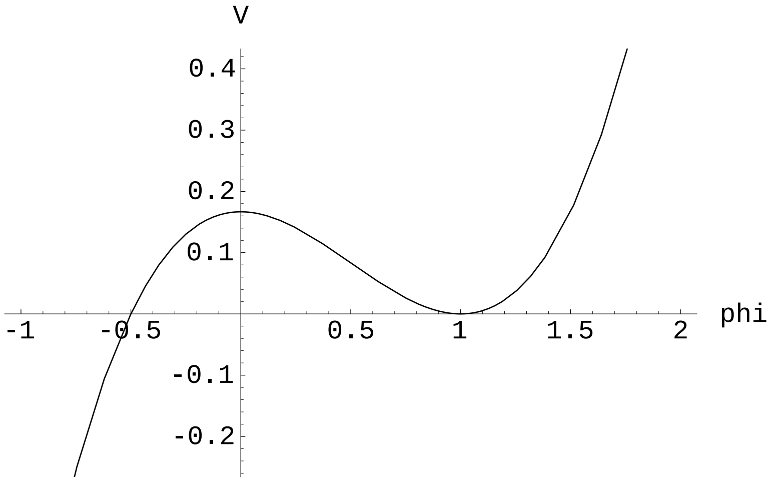

Figure 1: The cubic potential $V(\phi)$. The tachyonic vacuum is at $\phi = 0$ and the locally stable vacuum is at $\phi = 1$.

In the lump solution $\overline{\phi}(x)$, as $x \to \pm\infty$ we must have $\overline{\phi}(x) \to 1$, which corresponds to the locally stable vacuum. Given the familiar mechanical analogy of motion on the potential $(-V)$, we will find that $\overline{\phi}$ will vary from $+1$ down to $(-1/2)$ and back to $+1$ as x goes from minus to plus infinity. For symmetry, we take $\overline{\phi}(x = 0) = -1/2$. The equation one must solve is:

$$\frac{d^2\overline{\phi}}{dx^2} - V'(\overline{\phi}) = 0 \longrightarrow \frac{1}{2}\Big(\frac{d\overline{\phi}}{dx}\Big)^2 = V(\overline{\phi}) \longrightarrow x = \int_{-1/2}^{\overline{\phi}} \frac{d\phi'}{\sqrt{2V(\phi')}}\,. \tag{2.4}$$

With the potential in (2.3), the above integral is elementary[2] and gives the profile:

$$\overline{\phi}(x) = 1 - \frac{3}{2}\,\mathrm{sech}^2\left(\frac{x}{2}\right), \tag{2.5}$$

whose plot is shown in figure 2. We record, in passing, that

$$V''(\overline{\phi}) = 1 - 3\,\mathrm{sech}^2\left(\frac{x}{2}\right). \tag{2.6}$$

We now expand the action (2.3) around the lump solution by letting $\phi \to \overline{\phi} + \phi$, where, with a little abuse of notation we now use ϕ to denote the fluctuation field around the lump. We find (recall we have set $\phi_* = 1$)

$$\begin{aligned} S = \int d^{p+1}y dx\Big[&-\frac{1}{2}\Big(\frac{d\overline{\phi}}{dx}\Big)^2 - V(\overline{\phi}) - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \\ &- \frac{1}{2}\phi\left(-\frac{\partial^2\phi}{\partial x^2} + V''(\overline{\phi})\phi\right) - \frac{1}{3}\phi^3\Big]. \end{aligned} \tag{2.7}$$

[2]This is because two of the three possible zeroes in $V(\phi)$ coincide. If the lump was placed on a circle of finite radius, the relevant potential would have three different zeroes, and the integral would be expressed in terms of complete elliptic functions.

JHEP09(2000)028

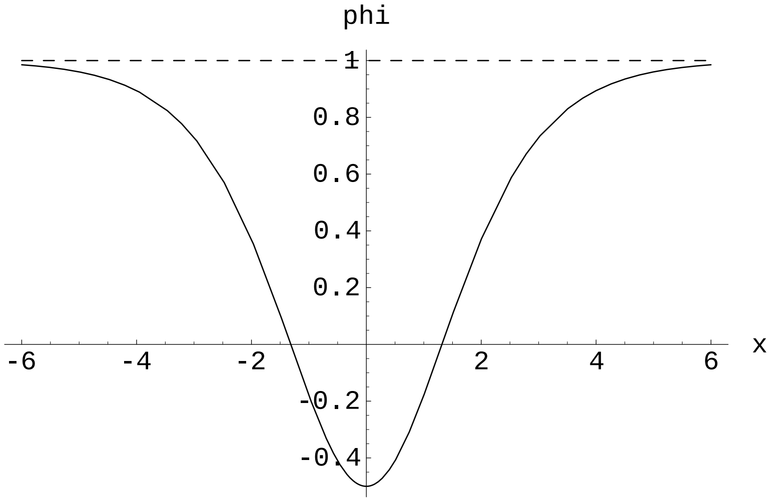

Figure 2: The profile of the lump solution $\overline{\phi}(x)$. As $x \to \pm\infty$, $\overline{\phi} \to +1$, which is the expectation value for the locally stable vacuum.

The first two terms of this action give (minus) the energy density of the p-brane defined by the lump solution. Indeed, the total energy E is given by

$$E = \int d^p\vec{y}dx \left[\frac{1}{2}\Big(\frac{d\overline{\phi}}{dx}\Big)^2 + V(\overline{\phi})\right] = (\mathrm{Vol}_y) \int dx \Big(\frac{d\overline{\phi}}{dx}\Big)^2 , \tag{2.8}$$

and performing the last integral one finds

$$T_p = \frac{E}{(\mathrm{Vol}_y)} = \frac{6}{5} , \tag{2.9}$$

for the tension of the p-brane. Given that the original ϕ^3 theory around the unstable vacuum $\phi = 0$ is supposed to represent the space-filling $(p+1)$-brane, we have $T_{p+1} = V(\phi = 0) = 1/6$. Therefore

$$\frac{1}{2\pi}\frac{T_p}{T_{p+1}} = \frac{18}{5\pi} \simeq 1.146 , \tag{2.10}$$

a ratio that in string theory takes the value of unity.[3]

The fluctuations on the brane require an analysis of the quadratic terms in (2.7). For this we consider the Schroedinger type eigenvalue equation

$$-\frac{d^2\psi_n}{dx^2} + V''(\overline{\phi}(x))\psi_n(x) = M_n^2\psi_n(x) . \tag{2.11}$$

The relevance of this equation is that expanding the fluctuation field $\phi(y, x)$ as

$$\phi(y, x) = \sum_n \xi_n(y)\psi_n(x) , \tag{2.12}$$

[3]In [10] the exact value of the string theory brane tension T_p is compared to the value T_p^ℓ obtained with the lump arising from the tachyon truncation of the string action. For this ratio we obtain the exact value $T_p^\ell/T_p = 8192\pi/32805 \simeq 0.7845$, in agreement with the numerical estimate in [10].

and substituting back in (2.7), the fields $\xi_n(y)$ living on the lump would be seen to have mass squared M_n^2 (this analysis will be done in section 3). Using the expression for $V''(\overline{\phi})$ in (2.6) we have

$$-\frac{d^2\psi_n}{dx^2}+\left(1-3\,\mathrm{sech}^2\left(\frac{x}{2}\right)\right)\psi_n(x)=M_n^2\psi_n(x)\,. \tag{2.13}$$

Letting $x\equiv 2u$ this equation is written as

$$-\frac{d^2\psi_n}{du^2}+(9-12\,\mathrm{sech}^2 u)\psi_n(u)=(4M_n^2+5)\psi_n(u)\,,\qquad x\equiv 2u\,. \tag{2.14}$$

This equation is the $\ell=3$ case of the following Schroedinger problem:

$$-\frac{d^2\psi_n}{du^2}+\left(\ell^2-\ell(\ell+1)\,\mathrm{sech}^2 u\right)\psi_n(u)=E_n(\ell)\psi_n(u)\,. \tag{2.15}$$

We now turn to the solution of this eigenvalue equation.

2.2 $\ell=3$ wavefunctions and lump spectrum

The eigenvalue equation (2.15) is readily solvable in terms of special functions. Indeed, it is a textbook case (see, for example [36, section 23, problem 5]) where the wavefunctions can be written in terms of associated Legendre polynomials when ℓ is an integer, and in terms of hypergeometric functions when it is not. The solution to be presented here (which I learned from J. Goldstone) uses methods explained in generality in [31]. One introduces a set of operators

$$a_\ell=\ell\tanh u+\frac{d}{du}\,,\qquad a_\ell^\dagger=\ell\tanh x-\frac{d}{du}\,, \tag{2.16}$$

and verifies that

$$H_\ell\equiv-\frac{d^2}{du^2}+\left(\ell^2-\ell(\ell+1)\,\mathrm{sech}^2 u\right)=a_\ell^\dagger a_\ell\,. \tag{2.17}$$

In addition, one readily confirms that:

$$H_{\ell-1}=a_\ell a_\ell^\dagger-(2\ell-1)\,. \tag{2.18}$$

The equation we must solve is:

$$H_\ell\psi_n^{(\ell)}(u)=E_n^{(\ell)}\psi_n^{(\ell)}(u)\,, \tag{2.19}$$

for the various eigenfunctions labelled by n, with $n=0$ denoting the ground state. The ground state wavefunction $\psi_0^{(\ell)}(u)$ is found from the condition

$$a_\ell\psi_0^{(\ell)}(u)=0\longrightarrow\psi_0^{(\ell)}(u)=N_0(\ell)\,\mathrm{sech}^\ell u\,. \tag{2.20}$$

The normalization of this wavefunction follows from the relations

$$I_\ell \equiv \int_{-\infty}^{\infty} du\, \mathrm{sech}^\ell u\,, \qquad I_{\ell+2} = \frac{\ell}{\ell+1} I_\ell\,, \qquad I_1 = \pi\,, \qquad I_2 = 2\,. \tag{2.21}$$

Thus, for example, for our $\ell = 3$ case of interest, we have

$$\psi_0^{(3)}(u) = \sqrt{\frac{15}{16}}\, \mathrm{sech}^3 u\,, \qquad E_0^{(3)} = 0\,. \tag{2.22}$$

To find the other wavefunctions, we use another feature of the hamiltonians H_ℓ: each $H_{\ell-1}$ eigenfunction provides an H_ℓ eigenfunction. Indeed, one readily verifies that

$$H_{\ell-1}\psi = E^{(\ell-1)}\psi \longrightarrow H_\ell(a_\ell^\dagger \psi) = (E^{(\ell-1)} + 2\ell - 1)(a_\ell^\dagger \psi) \equiv E^{(\ell)}(a_\ell^\dagger \psi)\,. \tag{2.23}$$

Applying this for $\ell = 3$, the ground state $\psi_0^{(2)} \sim \mathrm{sech}^2 u$ of H_2 is used to find

$$\psi_1^{(3)} \sim a_3^\dagger \psi_0^{(2)} \sim \tanh u\, \mathrm{sech}^2 u\,, \tag{2.24}$$

which upon normalization yields

$$\psi_1^{(3)}(u) = \sqrt{\frac{15}{4}} \tanh u\, \mathrm{sech}^2 u\,, \qquad E_1^{(3)} = 5\,. \tag{2.25}$$

Similarly, the ground state $\psi_0^{(1)} \sim \mathrm{sech}\, u$ of H_1 can be used to find, after two steps:

$$\psi_2^{(3)}(u) = \sqrt{\frac{3}{16}}\, (5\, \mathrm{sech}^3 u - 4\, \mathrm{sech}\, u)\,, \qquad E_2^{(3)} = 8\,. \tag{2.26}$$

The three states above are the only bound states. Their wave-functions are orthonormal, $\psi_0^{(3)}$ and $\psi_2^{(3)}$ are u even and $\psi_1^{(3)}$ is u odd. In addition to these bound states, we have a continuum of δ-function normalizable eigenfunctions. These can be found analogously starting with the δ-function normalizable continuum e^{iku} of $H_0 = -d/du^2$. We therefore have

$$\begin{aligned} \psi_k^{(3)}(u) &= \frac{1}{N(k)} a_3^\dagger a_2^\dagger a_1^\dagger e^{iku}\,, \quad N(k) = \sqrt{(1+k^2)(4+k^2)(9+k^2)}\,, \\ E_k &= 9 + k^2\,, \end{aligned} \tag{2.27}$$

satisfying the orthonormality condition

$$\int du \left(\psi_{k'}^{(3)}(u)\right)^* \psi_k^{(3)}(u) = 2\pi\delta(k - k')\,. \tag{2.28}$$

More explicitly, we have

$$\psi_k^{(3)}(u) = \frac{1}{N(k)} e^{iku} \Big[\tanh u \Big(6 - 6k^2 - 15\, \mathrm{sech}^2 u\Big) + ik\Big(k^2 - 11 + 15\, \mathrm{sech}^2 u\Big)\Big]\,. \tag{2.29}$$

The state at $k = 0$ is u odd, and while it is only δ-function normalizable, the wavefunction approaches a constant at infinity. It should be thought as a bound-state at threshold. It will play no special role in our discussion.

Having found all the relevant wavefunctions, we now summarize them and use them to provide the explicit solutions of our original equation (2.14). For this we use x rather than u, and give the corresponding mass squared values M_n^2 via the relation $E_n = 4M_n^2 + 5$. We also drop the $\ell = 3$ index, and use different symbols to denote x-even as opposed to x-odd wavefunctions:

$$\begin{aligned}
\xi_0(x) &= \sqrt{\frac{15}{32}}\,\mathrm{sech}^3\left(\frac{x}{2}\right), \qquad M^2 = -\frac{5}{4}\,, \\
\eta_0(x) &= \sqrt{\frac{15}{8}}\tanh\left(\frac{x}{2}\right)\mathrm{sech}^2\left(\frac{x}{2}\right), \qquad M^2 = 0\,, \\
\xi_1(x) &= \sqrt{\frac{3}{32}}\left(5\,\mathrm{sech}^3\left(\frac{x}{2}\right) - 4\,\mathrm{sech}\left(\frac{x}{2}\right)\right), \qquad M^2 = \frac{3}{4}\,.
\end{aligned} \tag{2.30}$$

We have one tachyon, one massless scalar and one massive scalar. While the original tachyon in the ϕ^3 theory had $m^2 = -1$, the tachyon living on the lump worldvolume has a larger mass $m^2 = -5/4$. In string theory both tachyons have the same mass. For the continuum, from (2.27) and (2.29), we introduce:

$$\begin{aligned}
\Xi_k(x) = \frac{e^{ikx/2}}{\sqrt{2}N(k)}\Big[&\tanh\left(\frac{x}{2}\right)\left(6 - 6k^2 - 15\,\mathrm{sech}^2\left(\frac{x}{2}\right)\right) + \\
&+ ik\left(k^2 - 11 + 15\,\mathrm{sech}^2\left(\frac{x}{2}\right)\right)\Big]\,,
\end{aligned} \tag{2.31}$$

with mass squared given by

$$M_k^2 = 1 + \frac{k^2}{4}\,. \tag{2.32}$$

The wavefunctions $\Xi_k(x)$ satisfy the reality and orthogonality properties

$$\Xi_k^*(x) = \Xi_{-k}(x)\,, \qquad \int dx\,\Xi_{k'}^*(x)\Xi_k(x) = 2\pi\delta(k - k')\,. \tag{2.33}$$

We introduce real and imaginary parts

$$\Xi_k(x) \equiv \eta_k(x) + i\xi_k(x)\,, \tag{2.34}$$

where

$$\begin{aligned}
\eta_{-k} &= \eta_k\,, \qquad \xi_{-k} = -\xi_k\,, \qquad \text{and} \\
\eta_k(-x) &= -\eta_k(x)\,, \qquad \xi_k(-x) = \xi_k(x)\,.
\end{aligned} \tag{2.35}$$

The wavefunctions ξ_k and η_k are associated with a continuum of scalar fields with mass squared greater than or equal to one.

JHEP09(2000)028

3. The action living on the ϕ^3 lump

In order to find the complete field theory living on the worldvolume of the lump we must expand the fluctuating field ϕ in terms of lump fields and lump wavefunctions, and substitute back into the action given in (2.7).

We begin by expanding the fluctuating field (as suggested in (2.12)) using the notation introduced above. This gives

$$\phi(y,x) = \phi_0(y)\xi_0(x) + \psi_0(y)\eta_0(x) + \phi_1(y)\xi_1(x) + \int_{-\infty}^{\infty} dk\Phi_k(y)\Xi_k(x)\,. \tag{3.1}$$

Here ϕ_0 is the tachyon, ψ_0 is the massless scalar and ϕ_1 is the massive scalar, all living on the lump worldvolume. Reality of $\phi(x,y)$, on account of (2.33), requires that

$$\begin{aligned} \Phi_k^*(y) &= \Phi_{-k}(y) \longrightarrow \Phi_k(y) \equiv \psi_k(y) + i\phi_k(y)\,, \quad \text{with} \\ \phi_{-k} &= -\phi_k\,, \qquad \psi_{-k} = \psi_k\,. \end{aligned} \tag{3.2}$$

We will focus our attention on the exact potential for the fields living on the lump. For this, it suffices to consider the part of the action in (2.7) that includes the last two terms:

$$V = \int d^{p+1}y \left[\int dx \left\{\frac{1}{2}\phi\Big(-\frac{d^2\phi}{dx^2} + V''(\overline{\phi})\phi\Big) + \frac{1}{3}\phi^3\right\}\right]. \tag{3.3}$$

In order to find the potential for the standard fields (as opposed to the continuum fields) we must simply substitute $\phi = \phi_0\xi_0(x) + \psi_0\eta_0(x) + \phi_1\xi_1(x)$ into the above, and perform the integral over x. The resulting potential takes the form $V = \int d^{p+1}yV_{\rm dis}$, with

$$\begin{aligned} V_{\rm dis} = & -\frac{5}{8}\phi_0^2 + \frac{3}{8}\phi_1^2 + \\ & + \frac{175}{8192}\sqrt{\frac{15}{2}}\pi\phi_0^3 + \frac{225}{8192}\sqrt{\frac{3}{2}}\pi\phi_0^2\phi_1 + \frac{129}{8192}\sqrt{\frac{15}{2}}\pi\phi_0\phi_1^2 - \\ & - \frac{201}{8192}\sqrt{\frac{3}{2}}\pi\phi_1^3 + \left(\frac{75}{2048}\sqrt{\frac{15}{2}}\pi\phi_0 - \frac{105}{2048}\sqrt{\frac{3}{2}}\pi\phi_1\right)\psi_0^2\,. \end{aligned} \tag{3.4}$$

Note the masslessness of ψ_0, expected as it represents the translation mode of the lump.

For the continuum states we can also determine the potential. We focus on the quadratic terms only. Making use of (2.32), (2.33), and (2.34), and writing $V = \int d^{p+1}yV_{\rm cont}^{(2)}$, we find

$$\begin{aligned} V_{\rm cont}^{(2)} &= \frac{1}{2}\int dx \int_{-\infty}^{\infty} dk'\Phi_{k'}(y)\Xi_{-k'}^*(x)\Big[-\frac{d^2}{dx^2} + V''(\overline{\phi})\Big]\int_{-\infty}^{\infty} dk\Phi_k(y)\Xi_k(x) \\ &= \pi\int_{-\infty}^{\infty} dk\Phi_{-k}(y)\Big(1 + \frac{k^2}{4}\Big)\Phi_k(y) \\ &= 2\pi\int_0^{\infty} dk\Big[\phi_k^2 + \psi_k^2\Big]\Big(1 + \frac{k^2}{4}\Big)\,. \end{aligned} \tag{3.5}$$

We can calculate all other interaction terms involving both discrete and continuum fields, as well as continuum fields only. Since our analysis will not make use of all of such terms and the expressions are considerably lengthy we will not attempt to record their explicit forms.

4. Analysis of tachyon condensation

In this section we begin by studying the decay of the lump with a level expansion analysis of the multiscalar tachyon potential. This is, to date, the only tool available in SFT. We then turn to an exact analysis, based on the fact that the vacuum state represents a well defined fluctuation of the lump profile. Finally, we discuss the flow of the masses of various states, including the tachyon, as the lump decays away.

4.1 Analysis in the level expansion

As is customary we will assign level zero to the tachyon field ϕ_0. Our first approximation to the condensation problem is to work at level zero, where we have the mass term and the cubic interaction of the tachyon ϕ_0 (see (3.4)):

$$V^{(0)}(\phi_0) = -\frac{5}{8}\phi_0^2 + \frac{175}{8192}\sqrt{\frac{15}{2}}\pi\phi_0^3\,. \tag{4.1}$$

The nontrivial critical point is at

$$\overline{\phi}_0 = \frac{2048}{105\pi}\sqrt{\frac{2}{15}} \simeq 2.26705\,, \tag{4.2}$$

giving us

$$V^{(0)}(\overline{\phi}_0) = -\frac{1048576}{99225\pi^2} \simeq -1.07073\,. \tag{4.3}$$

The absolute value of this result is our first approximation to the tension of the lump, given in (2.9). Thus forming the ratio, whose exact value should be unity, we find

$$\frac{|V^{(0)}(\overline{\phi}_0)|}{T_p} = \frac{524288}{59535\pi^2} \simeq 0.89227\,, \tag{4.4}$$

which at about 90%, is surprisingly close to the expected answer (in bosonic string theory this first approximation gives about 70% of the vacuum energy [7]).

To continue, recall that twist odd states play no role in tachyon condensation to the stable vacuum in open string theory. For completely analogous reason, states that arise from wavefunctions that are odd under $x \to -x$, such as ψ_0, will not acquire expectation values. So we can restrict ourselves to ϕ_0 and ϕ_1. Since the mass squared of ϕ_1 is $3/4$ and that of ϕ_0 is $(-5/4)$, the field ϕ_1 must be assigned level two $(= 3/4 - (-5/4))$.

We can therefore work out the (2,4) approximation (fields up to level two, and interactions up to level 4). This includes all terms involving ϕ_0 and ϕ_1 except for the term cubic in ϕ_1. This time we find

$$\text{Level (2,4)}: \ \overline{\phi}_0 \simeq 2.41575\,, \qquad \overline{\phi}_1 \simeq -0.43908\,, \tag{4.5}$$

giving us

$$V^{(2,4)}(\overline{\phi}) \simeq -1.1917 \longrightarrow \frac{|V^{(2,4)}(\overline{\phi})|}{T_p} \simeq 0.9931\,. \tag{4.6}$$

Indeed, this gives an extremely close value for the energy density of the lump. We now try the level (2,6) approximation, by including the ϕ_1^3 term. This gives

$$\text{Level (2,6)}: \ \overline{\phi}_0 \simeq 2.405\,, \qquad \overline{\phi}_1 \simeq -0.403234\,, \tag{4.7}$$

resulting in

$$V^{(2,6)}(\overline{\phi}) \simeq -1.1185 \longrightarrow \frac{|V^{(2,4)}(\overline{\phi})|}{T_p} \simeq 0.9872\,. \tag{4.8}$$

It may seem surprising that the error in the energy density has increased. This is presumably because once we are using level six interactions the continuum spectrum (whose fields start at level 9/4) could have played a significant role. At any rate we will see later on that the addition of the cubic term ϕ_1^3 has brought the expectation value of ϕ_1 significantly closer to its true value.

It is instructive to see how the above solutions are attempting to reconstruct the vacuum solution $\phi(x) = 1$. We can consider the function $C(x)$ defined as

$$C(x) = \overline{\phi}(x) + \overline{\phi}_0\xi_0(x) + \overline{\phi}_1\xi_1(x)\,. \tag{4.9}$$

This function is computed by adding to the profile $\overline{\phi}(x)$ the appropriately weighted wavefunctions of the first two states. As explained in the introduction, this sum evaluated at the exact expectation values, and including additionaly the continuum states should reproduce the vacuum configuration: $\overline{\phi}(x \to \infty) = 1$. Since we have only included the effect of the discrete fields, and their approximate expectation values $C(x)$ is only expected to be roughly equal to one. The plot of $C(x)$ for the various approximations we have taken is shown in figure 3. While the original profile $\overline{\phi}(x)$, shown in figure 2 extends from -0.5 to 1.0, we see that at level (2,4) (or (2,6)) $C(x)$ has already been flattened around one, extending less than 5% up or down.

4.2 Exact analysis of the condensation

The exact condensate is found from the condition that the fluctuation representing the condensate added to the profile $\overline{\phi}(x)$ must give the vacuum configuration $\phi(x)= 1$. Therefore we have the equation

$$1 - \overline{\phi}(x) = \frac{3}{2}\,\mathrm{sech}^2\left(\frac{x}{2}\right) = \overline{\phi}_0\xi_0(x) + \overline{\phi}_1\xi_1(x) - 2\int_0^\infty dk\overline{\phi}_k\xi_k(x)\,, \tag{4.10}$$

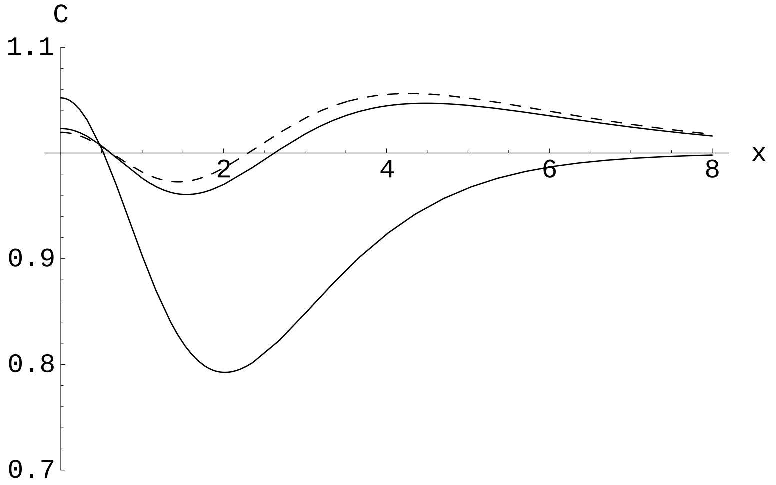

Figure 3: Plot of the function $C(x)$ defined in (4.9). At level zero we have the curve that extends down to about 0.8 (recall that the profile $\overline{\phi}$ went all the way down to -0.5). The level (2,4) curve is dashed, and the level (2,6) curve is continuous and close to the (2,4) curve.

where $\overline{\phi}_0, \overline{\phi}_1$, and $\overline{\phi}_k$ are the desired expectation values for the lump fields representing the stable vacuum as seen from the lump. Using the orthogonality of the associated ξ eigenfunctions we find:

$$\begin{aligned}
\overline{\phi}_0 &= \frac{9}{32}\sqrt{\frac{15}{2}}\pi \simeq 2.41976\,, \qquad \overline{\phi}_1 = -\frac{3}{8}\sqrt{\frac{3}{32}}\pi \simeq -0.3607\,, \\
\overline{\phi}_k &= \frac{3}{16\sqrt{2}}\frac{k^2}{\sinh(k\pi/2)}\sqrt{\frac{4+k^2}{(1+k^2)(9+k^2)}}\,.
\end{aligned} \tag{4.11}$$

Note that for the continuum states (2.32) the level L is given by $L(\overline{\phi}_k) = 1+k^2/4+5/4$ (see (2.32)). Taking the large k limit of the amplitude $\overline{\phi}_k$ we find

$$\overline{\phi}_k \sim \sqrt{L}\exp(-\pi\sqrt{L})\,. \tag{4.12}$$

Since all fields have properly normalized kinetic terms, this gives an idea of how fast the expectation values of fields decay as we go up in the level expansion. There is still no analogous result in the string field theory–in this case the number of states themselves increase as the level increases. In the present model we have two states at each continuum level, one arising from an even wavefunction, and one arising from an odd wavefunction.

We can do a consistency check by using the following simple result: given a multiscalar potential consisting of quadratic and cubic terms only, the quadratic terms evaluated at the solution give a value three times larger than the value of the potential at the solution. For our case, we must therefore have that the value $(-6/5)$

of the potential at the minimum can be written as

$$-\frac{6}{5} = \frac{1}{3}\left[-\frac{5}{8}\overline{\phi}_0^{\,2} + \frac{3}{8}\overline{\phi}_1^{\,2} + 2\pi\int_0^\infty dk\phi_k^2\Big(1+\frac{k^2}{4}\Big)\right], \tag{4.13}$$

where the terms inside brackets in the right hand side are the quadratic terms in the potential evaluated at the solution. Using the values quoted in (4.11) we must have

$$-\frac{6}{5} = -\frac{999\pi^2}{8192} + \frac{3\pi}{1024}\int_0^\infty dk\frac{k^4(4+k^2)^2}{(1+k^2)(9+k^2)\sinh^2(k\pi/2)}\,. \tag{4.14}$$

While the definite integral in the right hand side can surely be done analytically, the correctness of the above equation has been verified numerically to high accuracy. Equation (4.14) can be viewed as a kind of spectral decomposition of the lump energy. The continuum lump spectrum is seen to carry about 0.298% of the lump energy, a small part indeed.

4.3 Tracking the flow of the states

In this subsection we will study the fate of the fields living on the lump as the tachyon condenses. We will focus on the fields ϕ_0 and ϕ_1, namely, the tachyon and the massive scalar. We will imagine giving expectation values to the tachyon ϕ_0 that vary from zero to $\overline{\phi}_0$. As in the discussion of [35], we imagine adding a source term that cancels the tadpole arising because we are not at stationary points of the potential. For every value of ϕ_0 we solve for ϕ_1 using its equation of motion. Some intuition to the generic behavior of the flow follows from the discussion in [35]. One expects both fields to eventually merge into the continuum and there will be a particular configuration along the flow where the tachyon will have zero mass.

We consider first directly expanding around the tachyonic vacuum. To first approximation we can simply work to level zero, where $\overline{\phi}_0 = 2.267$ (see (4.2)). Expanding the level zero potential around this expectation value by letting $\phi_0 \to 2.267 + \phi_0$ we find that

$$V^{(0)} \simeq -1.071 + 0.625\phi_0^2 + \mathcal{O}(\phi_0^3)\,, \tag{4.15}$$

which implies a $m^2 \simeq 1.25$ for the tachyon after condensation.

Consideration of the next level of approximation shows this is not a very precise result. We expand the level (2,4) potential around the (2,4) expectation values given in (4.5) to find the quadratic form

$$V^{(2,4)} \simeq -1.192 + 0.661\phi_0^2 + 0.392\phi_0\phi_1 + 0.702\phi_1^2 + \mathcal{O}(\phi^3)\,. \tag{4.16}$$

After diagonalization one obtains two masses: $m^2 = 0.969, 1.756$. Finally, we can use the exact expectation values for the fields, as given in (4.11) and insert them into the level (2,6) potential finding the quadratic form

$$V^{(2,6)} \simeq -1.183 + 0.671\phi_0^2 + 0.413\phi_0\phi_1 + 0.805\phi_1^2 + \mathcal{O}(\phi^3)\,. \tag{4.17}$$

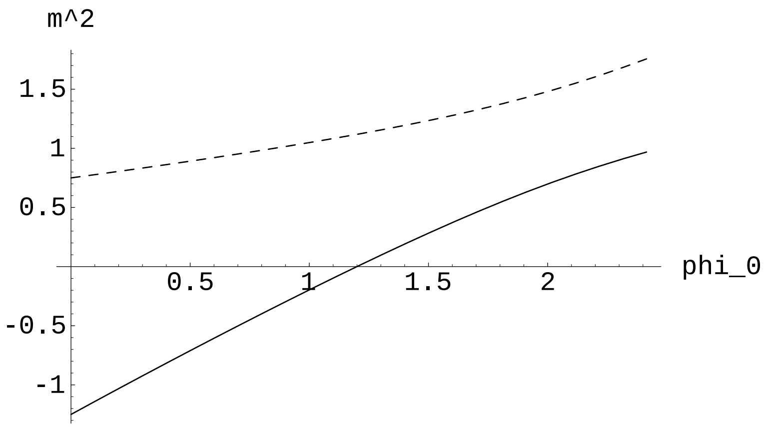

Figure 4: The solid curve shows the flow of the mass squared for the lowest mass state on the lump. The flow begins at $m^2 = -5/4$. The dashed line shows the flow of the massive state.

In this case after diagonalization we find the following eigenvalues and associated wave-functions (recall ϕ_0 and ϕ_1 are associated to $\xi_0(x)$ and $\xi_1(x)$):

$$\begin{aligned} m_1^2 &\simeq 1.04\,, \quad 0.81\xi_0 - 0.59\xi_1\,, \\ m_1^2 &\simeq 1.91\,, \quad 0.59\xi_0 + 0.81\xi_1\,. \end{aligned} \tag{4.18}$$

As we can see, the wavefunction of the first state is mostly ξ_0 and that of the second state is mostly ξ_1. We therefore expect the first state to be the endpoint of the flow of the tachyon, and the second state to be the endpoint of the flow of the massive state. Indeed, this way the two mass levels do not cross under the flow. In order to confirm this we have studied the flow of the masses by letting ϕ_0 vary, using the ϕ_1 equation of motion to fix ϕ_1 as a function of ϕ_0, and diagonalizing the mass matrix for every value of $\phi_0 \in [0, \overline{\phi}_0]$. The result is shown in figure 4.

Let us try to understand the meaning of what we have just calculated. The rearrangement of the lump degrees of freedom into the vacuum degrees of freedom is based on the simple statement that the plane waves of the vacuum can be expanded in terms of the lump wavefunctions as:

$$e^{ipx} = \phi_0(p)\xi_0(x) + \phi_1(p)\xi_1(x) + \psi_0(p)\eta_0(x) + \int dk \Phi_k(p)\Xi_k(x)\,. \tag{4.19}$$

In here, the expansion coefficients $\phi_0(p), \phi_1(p)$ and $\Phi_k(p)$ are readily calculable. As we let the lump begin to flow into the vacuum, we expect the potential to define a mass matrix that ceases to be diagonal and whose mass eigenstates become linear combinations of the lump wavefunctions. As we approach the vacuum, those linear combinations must approach the ones given in the above right hand side, for the pure exponentials are the eigenstates of the mass matrix around the vacuum. In fact, $m^2 = 1 + p^2$ for the wave $\exp(ipx)$.

It follows readily form (4.19) that

$$\phi_0(p) = \sqrt{\frac{15}{2}}\frac{\pi(p^2+1/4)}{\cosh p\pi}\,, \qquad \phi_1(p) = \sqrt{\frac{3}{2}}\frac{5\pi(p^2-3/20)}{\cosh p\pi}\,, \tag{4.20}$$

giving a ratio of

$$\frac{\phi_1(p)}{\phi_0(p)} = \sqrt{5}\,\frac{p^2-3/20}{p^2+1/4}\,. \tag{4.21}$$

We can now use the above result to test our flow. The first state of (4.18) has $m^2 = 1.04$ and $\phi_1/\phi_0 = -0.59/0.81 = -0.727$. Back in (4.21) this gives $p^2 = 0.051$ and a mass squared $m^2 = 1+p^2 \simeq 1.05$. This is in very good agreement with the value 1.04 found directly by diagonalization. For the other state, we have $m^2 = 1.91$ and $\phi_1/\phi_0 = 1.37$. Back in (4.21) this gives $p^2 \simeq 0.79$, and therefore predicts $m^2 = 1.79$, in reasonable agreement with the value 1.91 obtained by diagonalization.

Since momentum dependence plays no role in the field theory potential, as opposed to the case in string theory, the whole flow of masses is due to mass terms changing. In string theory both m^2 and p^2 terms change under the flow, and the resulting flow of masses (determined from zeroes in the inverse propagator p^2+m^2) will be a combined effect of these two changes. Certainly mass squared terms alone can do the job in the field theory case.

5. Large marginal deformations: moving the lump

In a recent work of Sen and the author [19] a puzzle was found. In order to study large marginal deformations the authors calculated in the level expansion the effective potential for the marginal parameter. The marginal parameter was taken to be the expectation value of the zero mode of the gauge field on a D-brane, a Wilson line. In the T-dual picture, this marginal parameter is the parameter translating the brane, and this viewpoint was emphasized in the early analysis of [8]. Surprisingly, it was found in [19] that there is a critical value of the string field marginal parameter beyond which its effective potential fails to exist. This raised a puzzle. In conformal theory both the Wilson line parameter or the translation parameter take values from zero to infinity. How does the string field theory manage to describe the physics with a finite range marginal parameter? Two options were discussed: the string field critical value corresponds to (i) infinite CFT parameter, or (ii) finite CFT parameter. The first possibility was considered to be better, the second worse, for it seemed to imply that SFT could not describe in a single patch all of CFT moduli space.

The interpretation here, to be substantiated below, is that indeed the critical value corresponds to a finite CFT parameter (possibility (ii)) but this *does not* mean that SFT does not cover CFT moduli space. What happens is that there is a double valued relation: to every SFT parameter we associate two CFT parameters. As the CFT parameter parameter grows from zero to infinity the SFT parameter grows from zero to a maximum value and then decreases back to zero! Of course, as the SFT marginal parameter starts to decrease fields higher in the level expansion must take larger expectation values.

To substantiate the above claim let us consider the lump field theory, where the field ψ_0 associated with the wavefunction $\eta_0(x)$ plays the role of a marginal parameter. Since η_0 is proportional to the derivative of the profile, ψ_0 is the parameter associated with translating the lump along the x coordinate. Thus, ψ_0 plays the role of the string field marginal parameter in the present model.

We begin by considering the potential (3.4) restricted to the tachyon and the marginal parameter:

$$V(\phi_0, \psi_0) = -\frac{5}{8}\phi_0^2 + \frac{175}{8192}\sqrt{\frac{15}{2}}\pi\phi_0^3 + \frac{75}{2048}\sqrt{\frac{15}{2}}\pi\phi_0\psi_0^2\,. \tag{5.1}$$

In order to find the effective potential for the marginal field we must use the tachyon equation to find $\phi_0(\psi_0)$:

$$\phi_0^{\pm}(\psi_0) = -\frac{1}{210\sqrt{30}\pi}\left(-4096 \pm \sqrt{16777216 - 756000\pi^2{\psi_0}^2}\right) \tag{5.2}$$

which indeed shows that the domain of definition of the effective potential for ψ_0 cannot exceed

$$|\psi_0| \leq \frac{512}{15\pi}\sqrt{\frac{2}{105}} \simeq 1.4995\,. \tag{5.3}$$

This is exactly parallel to the situation in string field theory [19]. There are two branches to the solution. In the notation of [19], the marginal branch is that where the tachyon begins with zero expectation value and corresponds to the top sign in (5.2). In the stable branch, corresponding to the bottom sign, the tachyon begins with its expectation value for the stable vacuum. The effective potential on the marginal branch is obtained substituting ϕ_0^+ into (5.1). Expanding the result for small ψ_0 we find

$$V_{\text{eff}}^M(\psi_0) = \frac{16875\pi^2}{4194304}{\psi_0}^4 + \mathcal{O}(\psi_0)^6\,. \tag{5.4}$$

As in [8, 19] this effective potential has a leading quartic term, but would be expected to be identically zero in the exact solution. For the stable branch we find:

$$V_{\text{eff}}^S(\psi_0) = -\frac{1048576}{99225\pi^2} + \frac{5{\psi_0}^2}{7} + \mathcal{O}(\psi_0)^4\,, \tag{5.5}$$

and we see that to this first rough approximation, we get $m^2(\psi_0) = 10/7 \simeq 1.429$ for the marginal mode around the stable vacuum. Including the effects of the field ϕ_1 changes the above results moderately. Working to level $(2,4)$ in these fields and exactly on ψ_0 we find that the domain of definition of the marginal effective potential shrinks down to about $|\psi_0| \leq 1.426$. Substituting the exact values $\overline{\phi}_0$ and $\overline{\phi}_1$ at the vacuum into (3.4) we can read a mass term for ψ_0 of

$$m^2(\psi_0) = \frac{5535\pi^2}{32768} \simeq 1.667\,. \tag{5.6}$$

In fact, the exact version of this result is readily obtained. We are simply studying the mass term associated to the fluctuation $\tilde{\phi} = \psi_0\eta_0(x)$ around the stable vacuum $\phi = 1$. The contribution to (minus) the action from this fluctuation is quadratic and given simply by $1/2 \int dx((\tilde{\phi}')^2 + (\tilde{\phi})^2)$. Numerical integration gives

$$m^2(\psi_0) = 1.714286\,, \tag{5.7}$$

for the mass of this mode around the stable vacuum. Note that the approximation with two fields in (5.6) was good. In string field theory a nonvanishing mass squared was found for the fluctuation mode of the marginal parameter around the stable. This was consistent with the expectation that the vacuum has no marginal deformations [19] and with the expectation that the gauge field on the brane dissappears after condensation [20].

We now turn to an explanation for the finite range of marginal parameters. As we have seen above, it happens in the field theory model for the "marginal" state ψ_0, so it is not a strictly stringy phenomenon. Consider the deformation that moves the lump a distance a to the left. From the viewpoint of the lump, this is a field fluctuation of the form $\overline{\phi}(x+a) - \overline{\phi}(x)$ since added to the lump profile $\overline{\phi}(x)$ it gives us $\overline{\phi}(x+a)$, representing the lump at $x = -a$. This fluctuation is therefore to be expanded as usual:

$$\overline{\phi}(x+a) - \overline{\phi}(x) = \phi_0(a)\xi_0(x) + \psi_0(a)\eta_0(x) + \cdots\,, \tag{5.8}$$

where the expansion coefficients are a-dependent. For the marginal mode one finds

$$\psi_0(a) = \int_{-\infty}^{\infty} dx \Big(\overline{\phi}(x+a) - \overline{\phi}(x)\Big)\eta_0(x) = \int_{-\infty}^{\infty} dx\overline{\phi}(x+a)\eta_0(x)\,, \tag{5.9}$$

where we used the $x \to -x$ symmetry of $\overline{\phi}(x)$ and antisymmetry of $\eta_0(x)$. This integral is readily done for small a. To this end first use (2.5) and (2.30) to obtain

$$\frac{d\overline{\phi}}{dx} = \sqrt{\frac{6}{5}}\eta_0(x)\,, \tag{5.10}$$

which gives the precise normalization relating the derivative of the profile to the massless mode representing the marginal operator. With this result, the integral (5.9) is readily done for small a giving:

$$\psi_0(a) = \sqrt{\frac{6}{5}}a + \mathcal{O}(a^2)\,. \tag{5.11}$$

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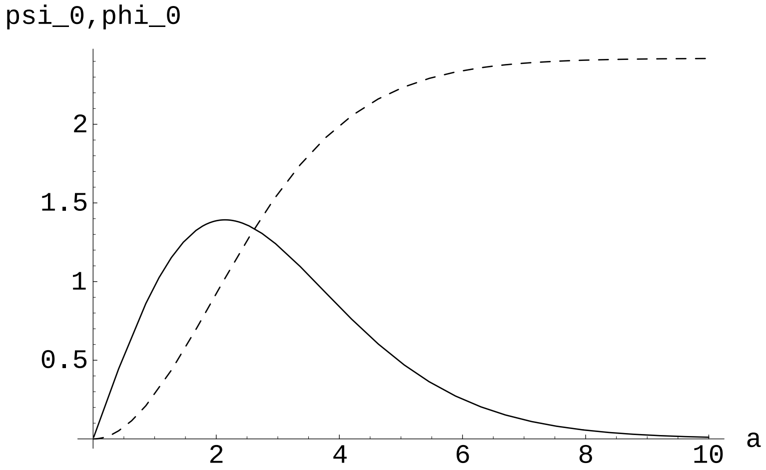

Figure 5: The solid curve shows the expectation value of the lump marginal parameter ψ_0 as a function of the displacement a of the lump. Note that the marginal parameter first increases, reaches a maximum, and then decreases. The dashed line shows the expectation value of the tachyon field ϕ_0 as the lump is displaced. Note that as the displacement is large the expectation value of ϕ_0 reaches the critical value $\overline{\phi}_0$ associated to the stable vacuum.

This is the expected linear relation between the "SFT marginal parameter" ψ_0 and the "CFT marginal parameter" a. On the other hand it is manifest from the integral expression (5.9) for $\psi_0(a)$ and the fact that $\eta_0(x)$ is localized in x, that for sufficiently large a the overlap between the lump, now centered at a and $\eta_0(x)$ will go to zero. Thus, very large a will correspond to small $\psi_0(a)$. This is the double valued relation we mentioned before. In figure 5 we show, in continuous line, the value of $\psi(a)$. Indeed, the plot begins straight but ψ_0 reaches a maximum value of $\psi_0^{\max} \simeq 1.39$ for $a \simeq 2.3$. This maximum value is in good agreement with the level (2,4) approximation that resulted in $|\psi_0| \leq 1.426$. For values of a larger than $a = 2.3$ the magnitude of ψ_0 decreases.

The expansion in (5.8) suggests what is happening when $a > 2.3$. In this case, some higher level fields, indicated by the dots will acquire larger expectation values. Thus for each value of ψ_0 we obtain two values of a, the small one realized with small expectation values for the high level fields, and the large one realized with large expectation values for some high level fields.

We now investigate the expectation value of the tachyon field $\phi_0(a)$ as a function of the displacement. In this case it follows from (5.8) that

$$\phi_0(a) = \int_{-\infty}^{\infty} dx \Big(\overline{\phi}(x+a) - \overline{\phi}(x)\Big)\xi_0(x)\,. \tag{5.12}$$

For small a the terms in parenthesis are well approximated by a times the derivative of the profile. Since ξ_0 is even, the integral cancels, and we must have $\phi_0(a) \sim a^2 + \mathcal{O}(a^4)$. On the other hand for large a the first term in the parenthesis becomes

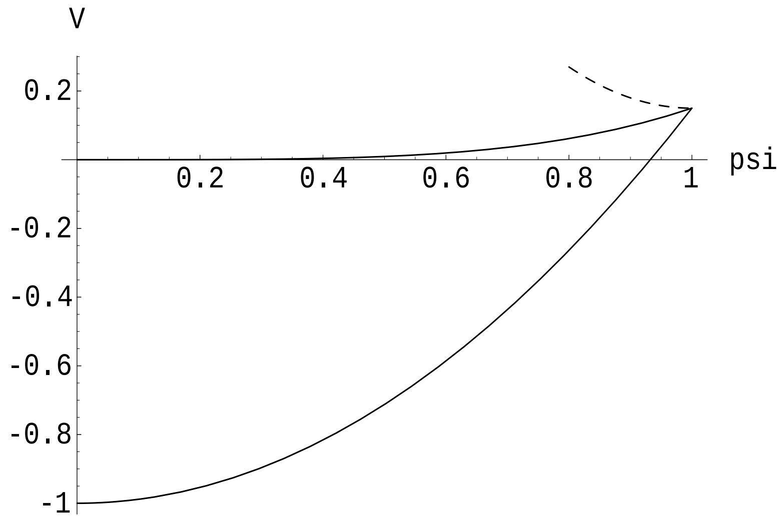

Figure 6: A sketch of various physical branches as the normalized marginal parameter ψ varies from zero to its maximal value. The vertical axis denotes the potential energy. The solid curve hugging the real line represents the marginal branch. It represents moving the lump all the way to $a \simeq 2.3$. The bottom curve (solid) represents the stable branch. The dashed curve, represents a missing branch describing displacements of the lump larger than $a = 2.3$.

irrelevant to the integral, and we simply get the overlap of the tachyon wavefunction with the profile. This is precisely given by $\overline{\phi}_0$ (recall (4.10)). Thus $\phi_0(a) \to \overline{\phi}_0$ as $a \to \infty$. This behavior is shown in figure 5, where the $\phi_0(a)$ is shown as a dashed line. The intuition behind this result is clear, by the time the lump is far away we have recovered the vacuum in the vicinity of $x = 0$. This requires giving the tachyon the expectation value $\overline{\phi}_0$.

We can now interpret more fully the branch structure found in [19], as exemplified by figure 1 of that work. For convenience we have included figure 6 with a sketch of the branches. We show the potential energy V, normalized to the energy of the lump, and ψ_0 normalized to its maximum value. The solid curves were discussed in [19], and here we have added a dashed curve. The solid curve closest to the real line represents the marginal branch. In perfect approximation it should be flat. When the maximal value of ψ_0 is attained it represents the lump displaced to some finite distance ($a = 2.3$ in the model). The bottom solid curve is the stable branch, at its leftmost point it intersects $V = -1$ and represents the vacuum. In the present model it is clear what that curve represents. As we go from the branch point downhill to the left we are simply letting the lump, centered at $a = 2.3$ decay away. The dashed line represents the branch where the lump is moved beyond $a = 2.3$. In this branch we expect the vev's of higher level fields to increase. This branch has not been identified yet in the string field theory, but the study of the field theory model suggests very strongly that it will be found there.

6. Conclusions and open questions

We considered ϕ^3 field theory and used the exact lump solution and the complete solvability of the fluctuation problem to produce a detailed model where the decay of an unstable lump is seen from the viewpoint of the field theory living on the lump itself.

There are many analogies with string field theory studies of D-brane annihilation. In particular, the multiscalar potential on the lump is nontrivial, and the vacuum state can be explored in the level expansion. Indeed, if did not have the exact expression for the tachyonic condensate in the model, we would have been hard pressed to find it from the explicit form of the cubic multiscalar potential. The problem becomes simple only with the realization that the multiscalar condensate is basically the expansion of the lump profile in terms of the lump wavefunctions. It may be possible to apply these lessons fruitfully to the study of the tachyon condensate in string field theory.

One of the deepest questions with regards to string field theory is whether or not the string field is big enough. Indeed, before tachyon condensation and brane annihilation was shown to be described correctly by string field theory, it was thought that the string field would not reach far enough to describe this non-perturbative vacuum. Now we know this is not the case. In the recent work of Sen and the author [19] the possibility resurfaced that maybe the string field is not big enough, as large marginal deformations could possibly be beyond reach of the string field. Based on similar circumstances in the toy model, we have been able to propose a natural resolution where a different branch of the solution space contains the large marginal deformations. It thus seems likely that the string field, after all, is big enough.

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